A PORTFOLIO SELECTION AND CAPITAL ASSET PRICING MODEL

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1. Introduction

Ever since the developments of portfolio theory in the 1950s and the capital asset pricing model in the 1960s, an impressive bulk of research has concerned the valuation of financial assets and the equilibrium of capital markets. As later recognized, whenever a capital market equilibrium model is based on the investors' preferences and the firms' features, a particular instance of a general valuation model is considered. As remarked by Costantinides (1989, p. 1), "the unifying theme is that the various … valuation theories are specializations of the fundamental valuation equation, which equates the price of a claim to the expectation of the product of the future payoff and the marginal rate of substitution of the representative investor".

Several model classes can be identified within that body of literature: the capital asset pricing model (CAPM), the intertemporal capital asset pricing model (ICAPM), the arbitrage pricing theories (APT), the consumption oriented capital asset pricing model (CCAPM). Although still used at an empirical stage owing to a stronger explanatory power (see Mankiw and Shapiro, 1986), the CAPM is not a general equilibrium model; the risk premia offered by all capital assets are ranked, yet the risk premium offered by the market as a whole is not explained. However, the more general models have proved much weaker than the CAPM when empirically tested. For instance, according to the CCAPM, the volatility of actual consumption is not consistent with actual risk premia so that an equity premium puzzle has arisen (see Burnside and McCurdy, 1992).

All of the CAPMs focus on the determination of a capital market equilibrium by solving a problem in which the investors' attitude toward risk and the investors' forecasts on balance sheets are inputs while capital asset prices and the stochastic properties of the rates of return are outputs. However, as explained by Farrell (1997) among others, financial analysis is concerned with a slightly different problem in which forecasts on balance sheets and actual capital asset prices are inputs while the stochastic properties of the rates of return are outputs. As a consequence, if a

capital market is not at equilibrium, financial analysis and hence portfolio selection could in principle not rest on such models. It is therefore unclear how valuation errors could be eliminated so as to attain the appropriate capital market equilibrium.

The aim of this paper is to improve the characterisation of a capital market within the CAPM without losing its simplicity and explanatory power. To highlight the peculiarities of our model, the main steps in the development of the standard CAPM are briefly reviewed.

In the seminal works by Markowitz (1952) and Tobin (1958) every investor is deemed as a price taker, while the means and covariances of the rates of return on available securities are picked up as inputs to portfolio selection. As Sharpe (1964), Lintner (1965), and Mossin (1966) extend the normative theory of portfolio selection as a positive theory of capital market equilibrium, capital asset prices become outputs. Note that the same set of forecasts on balance sheets can result in different sets of capital asset prices and hence in different sets of means and covariances of the rates of return. In the light of this, an educated investor must be able to perform a “what if” analysis and attach prices and statistics of the rates of return to any pair of values of the risk free interest rate and the price of risk. Unfortunately, no direction is available on how the investors’ forecasts on balance sheets can be included within the CAPM, although a hint is provided by Sharpe (1970, Appendix D) among others. Since in the CAPM the forecasts on the rates of return do not depend on actual capital asset prices and the forecasts on balance sheets, investors can not use the CAPM to compare different feasible capital market equilibria. Moreover, neither the CAPM can explain the risk premium offered by the market as a whole, nor it can result in convenient valuation formulas. According to Varian (1993), the CAPM is a demand side model, often complemented with the single factor model, a black box supply side model of how the rates of return are generated.

An attempt is made in this work so as to overcome those questions in some respects. The standard CAPM is extended so as to supply the price of each capital asset and explain the risk premium offered by the market as a whole. The resulting CAPM can be used to perform a ”what if” analysis and compare different capital market equilibria reflecting different values of the risk free interest rate and the price of risk. To achieve this, the demand and supply sides in a capital market are separately modelled. Moreover, the algorithm for portfolio selection is adapted and also made suited to heterogeneous expectations. When doing so, it is recognized that the forecasts on the rates of return do depend on both actual capital asset prices and the forecasts on balance sheets. As a consequence, portfolio selection can be performed also when actual capital asset prices are other than equilibrium ones. In the light of this, it can be stated that if there were a great deal of rational investors having homogeneous expectations on balance sheets and sticking to the

theory under scrutiny, the capital market would adjust to the appropriate equilibrium irrespective of noise.

To focus attention on the financial issues, a steady state economy is considered. Nonetheless, the resulting model is as general as a CCAPM. In our opinion, the latter is more of a theoretical interest than an operational use, since its inputs are very hard to obtain in a real and hence noisy setting. In contrast, our model is less demanding in this respect; if the steady state assumption were relaxed, it could stand as an amenable reference for market practitioners. However, the present version is preliminary and meant to be a stimulating conceptual tool to study a key feature of a fundamentally efficient capital market, i.e. the link between the stochastic properties of the investors' forecasts on balance sheets and the stochastic properties of the rates of return. As long as this point is not well comprehended by market practitioners, real capital markets are unlikely to be fundamentally efficient (evidence against fundamental efficiency is discussed by Shiller, 1989 and 2000).

The paper is organized as follows. The capital market model is introduced in Section 2, where two trade offs between average return and risk are considered, one on the demand side, the other on the supply side. The capital market equilibrium is studied in Section 3 by imposing that both trade offs are met. In particular, the expression of the risk premium offered by the market as a whole is determined and simple valuation formulas are derived. Moreover, attention is also devoted to the convergence to such a capital market equilibrium. Some remarks on portfolio selection are given in Section 4, where the case of heterogeneous expectations on balance sheets is taken into account. Conclusions appear in Section 5.

2. The representation of firms and investors

The microfoundations of the capital market model to follow include: 1) a risk free interest rate; 2) N investors, each having a given absolute risk aversion coefficient; 3) n firms, each having a given capital invested in a business with given stochastic properties. Since production is represented in a very compact form, our model can not be regarded as a general equilibrium one. Nonetheless, our approach carries over to models in which production is given a more detailed representation.

Consider a capital market on which n assets are traded. Taxes, transaction costs, and default risks are assumed away. Each investor is a price taker and eager for profit. Each capital asset is an all equity financed firm, which pays out the whole net income as a dividend. No investment is made so that each book value of equity remains constant as time t proceeds.

The book rates of return on those equities evolve as a gaussian white noise $R(t)$ with mean vector and variance-covariance matrix

$$\overline{R} = \begin{bmatrix} \overline{R}_1 \\ \overline{R}_2 \\ \dots \\ \overline{R}_n \end{bmatrix}, \quad S^2 = \begin{bmatrix} S_1^2 & S_{21} & \dots & S_{n1} \\ S_{21} & S_2^2 & \dots & S_{n2} \\ \dots & \dots & \dots & \dots \\ S_{n1} & S_{n2} & \dots & S_n^2 \end{bmatrix} \tag{1}$$

Let the *n* book values of equity be the entries of vector *k* and the *n* prices of equity at $t = 0$ be the entries of vector *p*. Each investor follows a buy and hold policy. Since only dividends are cashed, attention is restricted to dividends, whereas capital gains and losses are disregarded. The market rate of return on the *j*-th equity in the time interval $(t, t+1)$ is accordingly defined as

$$r_j(t) = \frac{R_j(t) k_j}{p_j} \tag{2}$$

As shown in Subsection 2.2, the resulting capital market equilibrium is consistent with (2). The rate $r_j(t)$, unknown at time *t*, becomes known at time $t+1$. Let the *n* market rates of return in the time interval $(t, t+1)$ be the entries of vector $r(t)$. In the light of (1) and (2), $r(t)$ is a gaussian white noise with mean vector and variance-covariance matrix

$$\overline{r} = \begin{bmatrix} \dfrac{\overline{R}_1 k_1}{p_1} \\ \dfrac{\overline{R}_2 k_2}{p_2} \\ \dots \\ \dfrac{\overline{R}_n k_n}{p_n} \end{bmatrix}, \quad \sigma^2 = \begin{bmatrix} \left(\dfrac{k_1 S_1}{p_1}\right)^2 & \dfrac{k_2 k_1 S_{21}}{p_2 p_1} & \dots & \dfrac{k_n k_1 S_{n1}}{p_n p_1} \\ \dfrac{k_2 k_1 S_{21}}{p_2 p_1} & \left(\dfrac{k_2 S_2}{p_2}\right)^2 & \dots & \dfrac{k_n k_2 S_{n2}}{p_n p_2} \\ \dots & \dots & \dots & \dots \\ \dfrac{k_n k_1 S_{n1}}{p_n p_1} & \dfrac{k_n k_2 S_{n2}}{p_n p_2} & \dots & \left(\dfrac{k_n S_n}{p_n}\right)^2 \end{bmatrix} \tag{3}$$

Remark 2.1. The equations (3) imply that

$$\frac{\overline{r}_j}{\sigma_j} = \frac{\overline{R}_j}{S_j}$$

i.e. that valuation does not alter the incipient average return-risk ratios (recall that $\overline{R}_j$, S_j are given by (1), while $\overline{r}_j$, σ_j are given by (3)). Moreover, the variance-covariance matrices in (1) and (3) yield the same correlation matrix; in other words, valuation does not alter the incipient correlations between rates of return.

A capital market equilibrium requires that two trade offs between average return and risk are met, one on the supply side, the other on the demand side. Both trade offs are introduced below.

2.1. The supply side trade off between average return and risk

Let $k_a = \sum_{j=1}^{n} k_j$ denote the book value of aggregate equities and $R_a(t) = \sum_{j=1}^{n} \frac{k_j R_j(t)}{k_a}$ represent the book rate of return on aggregate equities. In the light of (1), $R_a(t)$ is a gaussian white noise with mean and variance

$$\overline{R}_a = \sum_{j=1}^{n} \frac{k_j \overline{R}_j}{k_a}, \quad S_a^2 = \sum_{j=1}^{n}\sum_{l=1}^{n} \frac{k_j k_l S_{jl}}{k_a^2} \tag{4}$$

where $S_{jj} = S_j^2$.

Let $p_a = \sum_{j=1}^{n} p_j$ denote the price of aggregate equities at $t = 0$. Owing to (2) the market rate of return on aggregate equities in the time interval $(t, t+1)$ takes the form

$$r_a(t) = \sum_{j=1}^{n} \frac{k_j R_j(t)}{p_a} = \sum_{j=1}^{n} \frac{k_j R_j(t)}{k_a} \frac{k_a}{p_a} = \frac{R_a(t) k_a}{p_a}$$

The rate $r_a(t)$, unknown at time *t*, becomes known at time $t+1$. Recall that each investor follows a buy and hold policy so that attention is restricted to dividends, whereas capital gains and losses are disregarded. Since $R_a(t)$ is a gaussian white noise, $r_a(t)$ is a gaussian white noise with mean and variance

$$\overline{r}_a = \frac{k_a}{p_a} \overline{R}_a, \quad \sigma_a^2 = \left(\frac{k_a}{p_a}\right)^2 S_a^2 \tag{5}$$

Remark 2.2. The equations (5) imply that

$$\frac{\overline{r}_a}{\sigma_a} = \frac{\overline{R}_a}{S_a}$$

i.e. that market valuation does not alter the incipient average return-risk ratio. Let ρ_{ja} (P_{ja}) be the correlation between the market (book) rate of return on the *j*-th equity and the market (book) rate of return on aggregate equities. It is readily ascertained that[1] $\rho_{ja} = \mathrm{P}_{ja}$. Once again valuation does not alter the incipient correlations between rates of return. In the light of Remark 2.1 both results are not unexpected.

The equations (5) can be rewritten as

$$\bar{r}_a = \frac{\bar{R}_a}{S_a}\sigma_a \tag{6}$$

thus pointing out the relationship between the average return and risk of aggregate equities. To each pair $(\bar{r}_a, \sigma_a)$ there corresponds a different price of aggregate equities p_a. In other words, the book statistics $\bar{R}_a$, S_a are given, whereas the market statistics $\bar{r}_a, \sigma_a$ depend on p_a. Equation (6) represents the supply side trade off between average return and risk.

2.2. The demand side trade off between average return and risk

The equations (1) are known to investors so that expectations are homogeneous. Let $t = 0$. Since each investor conforms to a buy and hold perspective, he forms a portfolio once and for all. This implies that transactions at $t > 0$ are ruled out. Since no investment is made by any firm, individual saving is impossible. Therefore, wages, dividends and interests have to be consumed. Suppose that wages are certain and constant through time. Picking up a portfolio amounts then to fixing the stochastic features of future consumption.

Consider the *j*-th investor, who has a wealth w_j and earns a wage ω_j, certain and constant through time. His preferences as well as his attitude toward risk are represented by an additively separable exponential utility function with Arrow-Pratt measure of absolute risk aversion α_j (see Pratt, 1964 and Arrow, 1965). The investor attempts to form a portfolio so as to maximise the discounted value of the expected utility of consumption, namely

$$\max E_0\left\{\sum_{t=0}^{+\infty}\frac{-\exp\left[-\alpha_j\left(w_j r_P(t)+\omega_j\right)\right]}{\left(1+\beta_j\right)^{t+1}}\right\} = \max\left\{\sum_{t=0}^{+\infty}\frac{E_0\left\{-\exp\left[-\alpha_j\left(w_j r_P(t)+\omega_j\right)\right]\right\}}{\left(1+\beta_j\right)^{t+1}}\right\} =$$

$$\max\frac{E_0\left\{-\exp\left[-\alpha_j\left(w_j r_P(0)+\omega_j\right)\right]\right\}}{\beta_j} \tag{7}$$

where β_j is his rate of time preference and E_0 denotes an expected value based on the information available at time $t = 0$. Moreover, the portfolio rate of return r_P has mean $\bar{r}_P$ and standard deviation σ_P^2. If reference is made to the notion of certainty equivalent, the optimisation problem (7) can be rewritten as

$$\max w_j\bar{r}_P - 0.5\alpha_j w_j^2\sigma_P^2 \tag{8}$$

since the utility function is exponential and $\bar{r}_P(0)$ is a gaussian random variable. In the light of (8) it can be realized that the optimal portfolio must be mean-variance efficient. Owing to the one

fund theorem, the optimal portfolio, a blend of the optimal combination of risky assets and the risk free asset, lies on the efficient frontier

$$\bar{r}_P = i + \vartheta\sigma_P \tag{9}$$

where i is the risk free interest rate, known to investors and constant through time, and ϑ is the price per unit of risk borne. Since expectations are homogeneous, the optimal combination of risky assets is the same for all of the investors and equal to any fraction of aggregate equities. Moreover, also equation (9) is the same for all of the investors and is known as the capital market line, a very compact representation of the demand side trade off between average return and risk. For more details on the one fund theorem and the capital market line, see Constantinides and Malliaris (1995). As proved in the Appendix, the equilibrium value of ϑ is

$$\vartheta = \frac{k_a S_a}{\sum_{j=1}^{N} 1/\alpha_j} = \frac{\tilde{\alpha} k_a S_a}{N} \tag{10}$$

where N is the number of investors, α_j is the absolute risk aversion coefficient of the j-th investor, $\tilde{\alpha}$ is the harmonic mean of the α_j's. According to equation (10), the larger both S_a, the variance of the book rate of return on aggregate equities, and $\tilde{\alpha}$, the harmonic mean of the absolute risk aversion coefficients, the larger ϑ, the price of risk.

Remark 2.3. Let $t \neq 0$. It is readily realized that there is no difference between E_t and E_0 so that the above results remain unchanged as time proceeds. Therefore, the portfolio decisions taken at $t = 0$ remain unchanged, which is consistent with the very notion of a buy and hold policy.

3. Capital market equilibrium

Since investors do not know the α_j's whereas they can assess ϑ, reference is made to the price of risk ϑ when examining the capital market equilibrium. Owing to Remark 2.3, the results to follow hold irrespective of time *t*.

First consider the aggregate equities and define any fraction of aggregate equities as the market portfolio *m*. When the capital market is at equilibrium, the trade offs between average return and risk are met on both the supply and the demand side. Equating (6) and (9) yields

$$\bar{r}_m = \frac{i}{\bar{R}_a - \vartheta S_a}\,\bar{R}_a, \quad \sigma_m = \frac{i}{\bar{R}_a - \vartheta S_a} S_a \tag{11}$$

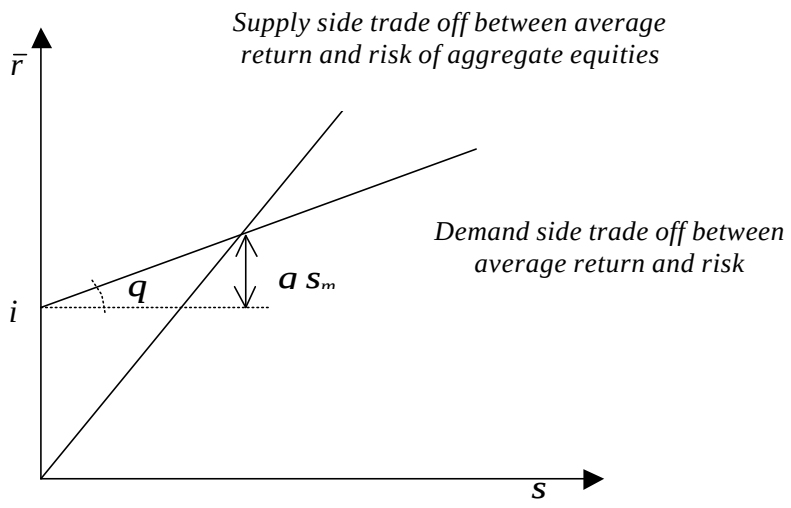

Figure 1 – Capital market equilibrium. The price of aggregate equities p_a increases when moving downward along the supply side trade off between average return and risk.

Owing to (10) and (11), the risk premium offered by the market as a whole takes the form

$$\vartheta\sigma_m = \vartheta\frac{i}{\bar{R}_a - \vartheta S_a} S_a = \frac{\tilde{\alpha} k_a S_a^2 i}{N\bar{R}_a - \tilde{\alpha} k_a S_a^2} \tag{12}$$

Now consider the *j*-th equity. Let *e* qualify any efficient portfolio. The correlation ρ_{je} between the market rate of return on the *j*-th equity and the market rate of return of portfolio *e* is given by

$$\rho_{je} = \frac{\bar{r}_j - i}{\sigma_j}\frac{\sigma_e}{\bar{r}_e - i}$$

namely by the ratio of the Sharpe's measure for the *j*-th equity to the Sharpe's measure of the portfolio *e* (see Costantinides and Malliaris, 1995, p. 14). Since the market portfolio *m* is efficient, using (9) within the previous equation yields

$$\bar{r}_j = i + \frac{(\bar{r}_m - i)}{\sigma_m}\rho_{jm}\sigma_j = i + \vartheta\rho_{jm}\sigma_j \tag{13}$$

Remark 3.1. Equation (13) defines the security market line. Note that the analytical derivation is other than that by Sharpe (1964). The term $\rho_{jm}\sigma_j$ is the systematic risk per unit of *j*-th equity. Recall that $p_m\sigma_m = \sum_{j=1}^{n} p_j\rho_{jm}\sigma_j$, i.e. the market risk of aggregate equities is the sum of the market systematic risks of the individual equities[2].

The equations (3) and (13) jointly entail

$$\bar{r}_j = \frac{i}{\bar{R}_j - \vartheta\rho_{jm}S_j}\bar{R}_j, \quad \sigma_j = \frac{i}{\bar{R}_j - \vartheta\rho_{jm}S_j}S_j \tag{14}$$

Recall that $R_a(t)k_a$ is the income provided by aggregate equities in the time interval $(t, t+1)$. Using (11) within (5) obtains

$$p_a = \frac{\bar{R}_a - \vartheta S_a}{i}k_a = \frac{\bar{u} - \vartheta S_u}{i} \tag{15}$$

where $\bar{u}$ and S_u are the mean and standard deviation of the income provided by aggregate equities. According to (15), the capital market equilibrium is such that the price of aggregate equities is equal to the certainty equivalent of aggregate income divided by the risk free interest rate.

Now recall that $R_j(t)k_j$ is the income provided by the *j*-th equity in the time interval $(t, t+1)$. Using (14) within (3) obtains

$$p_j = \frac{\bar{R}_j - \vartheta\rho_{jm}S_j}{i}k_j = \frac{\bar{u}_j - \vartheta\rho_{jm}S_{u_j}}{i} \tag{16}$$

where $\bar{u}_j$ and S_{u_j} are the mean and standard deviation of the income provided by the *j*-th equity. According to (16), the capital market equilibrium is such that the price of the *j*-th equity is equal to the certainty equivalent of income divided by the risk free interest rate.

Remark 3.2. Suppose that $\bar{R}_j - \vartheta\rho_{jm}S_j > i$. The equations (14) then imply that $\bar{r}_j < \bar{R}_j$ and $\sigma_j < S_j$. Moreover, equation (16) implies that $p_j > k_j$. In other words, if the average and risk adjusted book rate of return is larger than the risk free rate, price is larger than book value, thus reducing both the average market return and risk. Now suppose that $\bar{R}_j - \vartheta\rho_{jm}S_j < i$. The equations (14) then imply that $\bar{r}_j > \bar{R}_j$ and $\sigma_j > S_j$. Moreover, equation (16) implies that $p_j < k_j$. In other words, if the average and risk adjusted book rate of return is smaller than the risk free rate, price is smaller than book value, thus enhancing both the average market return and risk. Both results apply to aggregate equities as well. If *i* is too low (high), a great deal of capital asset prices are likely to be much larger (smaller) than their book values.

Remark 3.3. As for a "what if" analysis, an investor operating in the capital market described in Section 2 is thus endowed with the following tools:

- (11) and (15) to compute price and statistics of the market portfolio for any pair (i, ϑ);
- (14) and (16) to compute price and statistics of the *j*-th equity for any pair (i, ϑ).

Recall that the price of risk ϑ hinges on $\tilde{\alpha}$, the harmonic mean of the absolute risk aversion coefficients.

3. 1. Convergence to a capital market equilibrium

As explained by Black (1986), real capital markets are noisy. In other words, valuations errors do occur without giving rise to money machines. Suppose accordingly that valuation errors can occur in the capital market described in Section 2. In such a circumstance the means and covariances of the past market rates of return are unreliable, since future valuation errors are unlikely to be the same as past valuation errors. According to the received theory, portfolio optimisation would determine an optimal combination of risky assets different from the market portfolio. A great deal of investors would then attempt to adjust their portfolios accordingly. This would bring about a demand (supply) excess for underpriced (overpriced) equities. As a consequence, prices would be forced to change so as to attain an equilibrium. However, this explanation misses a key point. Since market data are unreliable owing to valuation errors, the output of portfolio optimisation is unreliable either. There is no reason to stick to it, which makes unclear how a capital market equilibrium could be attained. The flaw in the received theory is that it does not address the question of portfolio selection when actual prices are other than equilibrium ones.

As mentioned in the introduction, investors are able to form expectations on balance sheets rather than market rates of return. In our setting investors are assumed to know (1), i.e. the stochastic features underlying book data. Moreover, the market rates of return depend suitably on the book rates of return so that the computation of their statistics is straightforward, even if the capital market is not at equilibrium. As a consequence, when portfolio optimisation is based on such market rates, as outlined in Section 4, it provides a reliable output. If a great deal of investors stick to it, convergence to a capital market equilibrium is certain. This is in accordance with the procedure for financial analysis reported by Farrell (1997), which has been used in practice.

4. Portfolio selection

Consider an educated investor forming his portfolio at $t = 0$ and pursuing a buy and hold policy. He is obviously interested in determining the optimal combination of risky assets. Therefore, he faces the optimisation problem

$$\max_{x} \frac{\bar{r}_p - i}{\sigma_p} = \max_{x} \frac{\sum_{j=1}^{n} x_j \left(\bar{r}_j - i\right)}{\sqrt{\sum_{j=1}^{n}\sum_{l=1}^{n} x_j x_l \sigma_{jl}}} \tag{17}$$

where x_j, the *j*-th entry of vector *x*, is the proportion invested in the *j*-th equity $\left(\sum_{j=1}^{n} x_j = 1\right)$ while $\bar{r}_j$ and σ_{jl} are given by (3).

Remark 4.1. To solve problem (17), the investor can apply a well known optimisation procedure, presented, for instance, in Luenberger (1998, chap. 6). If prices are other than the equilibrium ones, it can be safely expected that the optimal combination of risky assets is different from the market portfolio, overweighing underpriced equities and underweighing overpriced equities. If the investor is able to guess the equilibrium value of ϑ, he can use (16) to compute equilibrium prices. Capital gains and losses implicit in the optimal combination of risky assets given by (17) can then be appraised by comparing actual prices with equilibrium ones.

Suppose that the equations (1) are not known to investors and that expectations on balance sheets are heterogeneous. Consider an educated individual forming his portfolio at $t = 0$ and pursuing a buy and hold policy. To determine the optimal combination of risky assets, he has still to solve the optimisation problem (17) introduced above, where the mean $\bar{r}_j$ and the covariance σ_{jl} now depend on the actual prices p_j, p_l as well as the investor's estimates of $\bar{R}_j$ and S_{jl}.

5. Conclusions

The CAPM put forward in this paper shares the merits of both the CCAPM and the standard CAPM. As a matter of fact, it rests on sound microeconomic foundations as a CCAPM: account is taken of such exogeneous components to a financial market as the firms' features, the risk free rate of return, and the individual attitudes toward return and risk, summarized by the harmonic mean of the absolute risk aversion coefficients. Moreover, our CAPM is able to represent the working of a financial market in a simple and straightforward fashion as a standard CAPM: some well known results concerning return and risk are more readily derived and new results about pricing are achieved. As a consequence, the explanatory power of a standard CAPM should be retained.

In spite of the above mentioned similarities, the line of reasoning is considerably different. The standard CAPM is a demand side model, often complemented with a black box supply side model

of how the rates of return are generated, whereas our model takes both demand and supply into account: two trade offs between return and risk are met when a capital market is at equilibrium. As a consequence, the usual procedure is reversed: first the information about aggregate equities has to be determined, then the case of a particular equity can be considered. Future applications can be based on our results. As previously shown, portfolio selection can be carried out under market inefficiences and heterogeneous expectations; in the former case, one can also calculate the overall capital gain accruing as equilibrium is attained. Moreover, the link established between book and market rates of return paves the way for the design of new tests about the fundamental efficiency of a capital market.

However, the above ideas can not be used in practice, unless our model is further extended. Although the reference to CARA utility functions was due to their analytical tractability, the use of CRRA utility functions would not alter our results. Neither would the allowance for net income retention: if an investor follows a buy and hold policy, he can assume a price dynamics (i.e. an amortisation plan) irrespective of whether it is met in practice. In that circumstance the price vector would behave as a random walk. It is well known that this does not come close to reality (see Lo and MacKinlay, 1999 on this subject). Nonetheless, we believe that a comparison of market rates of return with book rates of return would be meaningful for those investors who do not like to revise frequently their portfolios. For instance, if the standard deviation of a given market rate of return were above average, one could wonder whether this matches the statistics of the corresponding book rate of return.

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Appendix

Consider the j-th investor. Owing to the one fund theorem and the assumption of homogeneous expectations, the optimisation problem (8) specializes as follows

$$\max w_j \left[\left(1 - x_j\right) i + x_j \bar{r}_m \right] - 0.5\alpha_j w_j^2 x_j^2 \sigma_m^2 \tag{A1}$$

where the market portfolio *m* is defined by any fraction of the aggregate equities and x_j is the proportion of wealth w_j invested in the market portfolio. The optimal solution to (A1) is given by

$$x_j^* = \frac{\bar{r}_m - i}{\alpha_j w_j \sigma_m^2} \tag{A2}$$

If $1 - x_j^* < 0$, the j-th investor is indebted. The aggregate investment in risky assets must be the same as the price of aggregate equities so that

$$\sum_{j=1}^{N} x_j^* w_j = \frac{\bar{R}_a k_a}{\bar{r}_m}$$

Using (A2), (6) and (9) within the previous equation yields

$$\sum_{j=1}^{N} \frac{\vartheta}{\alpha_j} = S_a k_a$$

and hence the equation (10).

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Notes

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[1] Let $n = 2$. The extension to the case $n > 2$ can be obtained by following the same line of reasoning. Using (3) and (5) within the definition of ρ_{ja}

$$\rho_{ja} = \left(\frac{p_1}{p_a} \sigma_{j1} + \frac{p_2}{p_a} \sigma_{j2} \right) / \left(\sigma_j \sigma_a \right)$$

obtains the definition of P_{ja}

$$\rho_{ja} = \left(\frac{p_1}{p_a} \frac{k_j k_1}{p_j p_1} S_{j1} + \frac{p_2}{p_a} \frac{k_j k_2}{p_j p_2} S_{j2} \right) / \left(\frac{k_j k_a}{p_j p_a} S_j S_a \right) =$$

$$= \left(\frac{k_a}{k_a} \frac{k_j k_1}{p_j p_a} S_{j1} + \frac{k_a}{k_a} \frac{k_j k_2}{p_j p_a} S_{j2} \right) / \left(\frac{k_j k_a}{p_j p_a} S_j S_a \right) = \left(\frac{k_1}{k_a} S_{j1} + \frac{k_2}{k_a} S_{j2} \right) / \left(S_j S_a \right) = \mathrm{P}_{ja}$$

[2] Substituting (9) and (13) within the definition $p_m \bar{r}_m = \sum_{j=1}^{n} p_j \bar{r}_j$ yields $p_m \left(i + \vartheta \sigma_m \right) = \sum_{j=1}^{n} p_j \left(i + \vartheta \rho_{jm} \sigma_m \right)$ and hence the equation under scrutiny.